**Lesson 25: Logarithmic Functions**

Recall, that the exponential function $f(x) = a^x$ is a one-to-one function and so it has an inverse function which will be called the **logarithmic function**.

**Definition 1.** **Logarithmic Function:** *For $x > 0$, $a > 0$ and $a \neq 1$,*

$$y = \log_a x \quad \textit{if and only if} \quad x = a^y.$$

*The function $f(x) = \log_a x$ is called the **logarithmic function with base** $a$. Its **domain** is $(0, \infty)$.*

The definition of the logarithm says that the two equations

$$y = \log_a x \quad \textbf{(Logarithmic Form)}$$

$$x = a^y \quad \textbf{(Exponential Form)}$$

are equivalent. For example, $\log_3 81 = 4$ because 3 must be raised to the fourth power to obtain 81.

**Example 1.** **Converting from Exponential to Logarithmic Form:** *Write each exponential equation in logarithmic form.*

**a.** $4^3 = 64$ **b.** $(\frac{1}{2})^4 = \frac{1}{16}$ **c.** $a^{-2} = 7$

***Solution:***

**a.** $4^3 = 64$ *is equivalent to* $\log_4 64 = 3$.

**b.** $(\frac{1}{2})^4 = \frac{1}{16}$ *is equivalent to* $\log_{1/2} \frac{1}{16} = 4$.

**c.** $a^{-2} = 7$ *is equivalent to* $\log_a 7 = -2$.

**Example 2.** **Converting from Logarithmic Form to Exponential Form:** *Write each logarithmic equation in exponential form.*

**a.** $\log_3 243 = 5$ **b.** $\log_2 5 = x$ **c.** $\log_a N = x$

***Solution:***

**a.** $\log_3 243 = 5$ *is equivalent to* $243 = 3^5$.

**b.** $\log_2 5 = x$ *is equivalent to* $5 = 2^x$.

**c.** $\log_a N = x$ *is equivalent to* $N = a^x$.

**Example 3.** *Solve the following equation* $\log_2(x^2 - 6x + 10) = 1$.

***Solution:*** *To solve logarithmic equations we need to convert into exponential ones.*

$$\begin{aligned}\log_2(x^2 - 6x + 10) &= 1\\ x^2 - 6x + 10 &= 2^1 = 2\\ x^2 - 6x + 8 &= 0\\ (x-4)(x-2) &= 0.\end{aligned}$$

*Then* $x = 4$ *or* $x = 2$.

**Example 4.** *Find the domain of* $f(x) = \log_3(2 - x)$.

***Solution:*** *Because the domain of the basic logarithmic function* $y = \log_3 x$ *is* $(0, \infty)$, *the expression* $2 - x$ *must be positive. Then the domain of* $f(x)$ *is all real numbers such that:*

$$2 - x > 0,$$

*which means that* $x < 2$. *Hence, the domain of* $f(x) = \log_3(2 - x)$ *is* $(-\infty, 2)$.

**Definition 2. Basic Properties of Logarithms:** *For any base* $a > 0$, *with* $a \neq 1$,

1. $\log_a a = 1$.
2. $\log_a 1 = 0$.
3. $\log_a a^x = x$ *for any real number* $x$.
4. $a^{\log_a x} = x$ *for any* $x > 0$.

**Definition 3. Common Logarithms:** *The logarithm with* ***base*** 10 *is called the* ***common logarithm*** *and is denoted by omitting the base, so*

$$\log x = \log_{10} x.$$

*This means that for any* $x > 0$,

$$y = \log x \quad \text{if and only if} \quad x = 10^y.$$

**Definition 4. Natural Logarithms:** *The logarithm with* ***base*** $e$ *is called the* ***natural logarithm*** *and is denoted by*

$$\ln x = \log_e x.$$

*This means that for any* $x > 0$,

$$y = \ln x \quad \text{if and only if} \quad x = e^y.$$

**Example 5.** *Solve for the variable* $M$ *in the equation* $T = \log(7M - 3)$.

***Solution:*** *To solve logarithmic equation we need to convert into exponential one:*

$$\begin{aligned}T &= \log(7M - 3)\\ 10^T &= 7M - 3\\ 10^T + 3 &= 7M\end{aligned}$$

*Then* $M = \frac{10^T + 3}{7}$.